

Stochastic modelling of intermittent scrape-off layer plasma fluctuations

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Single-point measurements of fluctuations in the scrape-off layer of magnetized plasmas are generally found to be dominated by large-amplitude bursts which are associated with radial motion of blob-like structures. A stochastic model for these fluctuations is presented, with the plasma density given by a random sequence of bursts with a fixed wave form. Under very general conditions, this model predicts a parabolic relation between the skewness and kurtosis moments of the plasma fluctuations. In the case of exponentially distributed burst amplitudes and waiting times, the probability density function for the fluctuation amplitudes is shown to be a Gamma distribution with the scale parameter given by the average burst amplitude and the shape parameter given by the ratio of the burst duration and waiting times.

Cross-field transport of particles and heat in the scrape-off layer (SOL) of non-uniformly magnetized plasmas is caused by radial motion of blob-like structures [1–6]. This results in single-point recordings dominated by large-amplitude bursts, which have an asymmetric wave form with a fast rise and a slow decay, and positively skewed and flattened amplitude probability density functions [7–14]. Measurements on a number of tokamak experiments have demonstrated that as the empirical discharge density is approached, the radial SOL particle density profile becomes broader and plasma–wall interactions increase [11–15].

Probe measurements on Tokamak à Configuration Variable (TCV) have demonstrated a remarkable degree of universality of the plasma fluctuations in the far SOL region, which is dominated by radial motion of filament structures and a relative fluctuation level of order unity [10–13]. In particular, the amplitude distribution of the plasma fluctuations are found to be well described by a Gamma distribution across a broad range of plasma parameters and for all radial positions in the SOL [12]. Excellent agreement was found when comparing analysis of these data with turbulence simulations based on interchange motions [9–11].

In this Letter, a stochastic model for intermittent fluctuations in the plasma SOL is presented

with all statistical properties in agreement with experimental measurements. It is demonstrated that this model explains many of the salient experimental findings and empirical scaling relations, including broad plasma profiles and large fluctuation levels, skewed and flattened amplitude probability distribution functions and a parabolic relation between the skewness and kurtosis moments. The latter has been observed in the boundary region of numerous experiments on magnetized plasmas as well as in hydrodynamical and astrophysical systems dominated by intermittent fluctuations [12, 16–18].

There have been several previous attempts at describing the universal features of intermittent fluctuations at the boundary of magnetically confined plasmas [19–21]. However, none of these models provide the appealing simplicity, physical insight, novel predictions and favourable comparison with experimental measurements as the theory presented here. In particular, the statistical properties which the present model is based on have been directly confirmed by experiments [8, 16].

Experimental measurements as well as numerical simulations suggest that plasma fluctuations in the far SOL can be represented as a random sequence of bursts events,

$$\Phi(t) = \sum_k A_k \psi(t - t_k) \quad (1)$$

where A_k is the amplitude and t_k is the arrival time for burst event k , and ψ is a fixed burst wave form. This stochastic process resembles a general class of models known as “shot noise”, in which the noise is generated by the addition of a large number of disturbances [22–24]. The objective is to estimate the mean value and higher order moments of Φ , the amplitude probability density function P_Φ and discuss how the burst statistics are related to broad SOL plasma profiles and large fluctuation levels.

If there are K burst events in a time interval T , the average burst waiting time τ_w is given by T/K . It follows that the mean value of the plasma density is [22–24],

$$\langle \Phi \rangle = \frac{\langle A \rangle}{\tau_w} \int_{-\infty}^{\infty} dt \psi(t). \quad (2)$$

Here and in the following, angular brackets are defined as an average of a random variable over all its values. The above equation shows that the mean plasma density is given by the average burst amplitude and the ratio of the burst duration and waiting times. Equation (2) thus elucidates the role of burst statistics for high plasma density in the far SOL. It should be noted that this result only depends on the integrated burst wave form and the average burst amplitude and waiting time.

Expressions for the variance and higher order moments of Φ have been derived in the case that burst events occur in accordance to a Poisson process with rate $1/\tau_w$. The probability of exactly K burst events in a time interval T is then given by the Poisson distribution,

$$P(K) = \exp\left(-\frac{T}{\tau_w}\right) \left(\frac{T}{\tau_w}\right)^K \frac{1}{K!}. \quad (3)$$

From this it follows that the burst waiting times are exponentially distributed, as found from experimental measurements [8, 16],

$$P_\tau(\tau) = \frac{1}{\tau_w} \exp\left(-\frac{\tau}{\tau_w}\right). \quad (4)$$

The general result states that for the stochastic process defined by Eq. (1), the cumulants κ_n for the probability density $P_\Phi(\Phi)$ are given by [23, 24]

$$\kappa_n = \frac{\langle A^n \rangle I_n}{\tau_w}, \quad (5)$$

where the integral of the n -th power of the wave form is defined by

$$I_n = \int_{-\infty}^{\infty} dt [\psi(t)]^n. \quad (6)$$

The cumulants are the coefficients in the expansion of the logarithm of the characteristic function for P_Φ ,

$$\ln \langle \exp(i\Phi u) \rangle = \sum_{n=1}^{\infty} \kappa_n \frac{(iu)^n}{n!}. \quad (7)$$

A power series expansion shows that the characteristic function is related to the raw moments of Φ , defined by $\mu'_n = \langle \Phi^n \rangle$,

$$\langle \exp(i\Phi u) \rangle = 1 + \sum_{n=1}^{\infty} \frac{\langle i\Phi u \rangle^n}{n!} = 1 + \sum_{n=1}^{\infty} \mu'_n \frac{(iu)^n}{n!}. \quad (8)$$

Further expanding the logarithmic function in Eq. (7) and using Eq. (8), it follows that the lowest order centred moments $\mu_n = \langle (\Phi - \langle \Phi \rangle)^n \rangle$ are related to the cumulants by the relations $\mu_2 = \kappa_2$, $\mu_3 = \kappa_3$ and $\mu_4 = \kappa_4 + 3\kappa_2^2$.

The variance and higher order moments are straight forward to calculate from Eq. (5) for general burst wave forms and amplitude distributions. The coefficient of variation, skewness and

flatness are given respectively by

$$C = \frac{\langle (\Phi - \langle \Phi \rangle)^2 \rangle^{1/2}}{\langle \Phi \rangle} = \tau_w^{1/2} \frac{I_2^{1/2}}{I_1} \frac{\langle A^2 \rangle^{1/2}}{\langle A \rangle}, \quad (9a)$$

$$S = \frac{\langle (\Phi - \langle \Phi \rangle)^3 \rangle}{\Phi_{\text{rms}}^3} = \tau_w^{1/2} \frac{I_3}{I_2^{3/2}} \frac{\langle A^3 \rangle}{\langle A^2 \rangle^{3/2}}, \quad (9b)$$

$$F = \frac{\langle (\Phi - \langle \Phi \rangle)^4 \rangle}{\Phi_{\text{rms}}^4} = 3 + \tau_w \frac{I_4}{I_2^2} \frac{\langle A^4 \rangle}{\langle A^2 \rangle^2}. \quad (9c)$$

The two latter relations imply that there is a parabolic relation between the skewness and flatness moments,

$$F = 3 + \frac{I_2 I_4}{I_3^2} \frac{\langle A^2 \rangle \langle A^4 \rangle}{\langle A^3 \rangle^2} S^2. \quad (10)$$

Such a parabolic relation between the third and fourth order moments have been found for a wide variety of physical systems dominated by intermittent fluctuations [12, 16–21].

The expressions for the higher order moments become particularly simple for a burst wave form given by a sharp rise followed by a slow exponential decay,

$$\psi(t) = \Theta(t) \exp\left(-\frac{t}{\tau_d}\right), \quad (11)$$

where Θ is the step function and τ_d is the burst duration time. This is the typical wave form found from probe and gas puff imaging measurements in the far SOL [7–13]. The integral given in Eq. (6) is then $I_n = \tau_d/n$ and the cumulants are thus given by $\kappa_n = \tau_d \langle A^n \rangle / n \tau_w$. The expressions for the coefficient of variation, skewness and flatness become

$$C = \left(\frac{\tau_w}{2\tau_d}\right)^{1/2} \frac{\langle A^2 \rangle^{1/2}}{\langle A \rangle}, \quad (12a)$$

$$S = \left(\frac{8\tau_w}{9\tau_d}\right)^{1/2} \frac{\langle A^3 \rangle}{\langle A^2 \rangle^{3/2}}, \quad (12b)$$

$$F = 3 + \frac{\tau_w}{\tau_d} \frac{\langle A^4 \rangle}{\langle A^2 \rangle^2}. \quad (12c)$$

The relation between the skewness and flatness is in this case given by

$$F = 3 + \frac{9}{8} \frac{\langle A^2 \rangle \langle A^4 \rangle}{\langle A^3 \rangle^2} S^2. \quad (13)$$

Note that independent of the burst wave form, the probability distribution function for Φ is positively skewed, $S > 0$, and flattened, $F > 3$, for positive definite burst amplitudes A .

The above expressions for the lowest order moments simplify further in the case of exponentially distributed burst amplitudes,

$$P_A(A) = \frac{1}{\langle A \rangle} \exp\left(-\frac{A}{\langle A \rangle}\right). \quad (14)$$

which is also consistent with experimental measurements in the SOL of magnetically confined plasmas [8, 16]. The raw amplitude moments are then given by $\langle A^n \rangle = \langle A \rangle^n n!$. In this case the relative fluctuation level, skewness and flatness can be written as

$$C = \left(\frac{\tau_w}{\tau_d}\right)^{1/2} \quad S = \left(\frac{4\tau_w}{\tau_d}\right)^{1/2}, \quad F = 3 + \frac{6\tau_w}{\tau_d}. \quad (15)$$

All these moments increase with the ratio τ_w/τ_d . The parameter $\gamma = \tau_d/\tau_w$ is thus a measure of intermittency in the shot noise process. The relation between the skewness and flatness moments now becomes

$$F = 3 + \frac{3}{2} S^2, \quad (16)$$

which is in excellent agreement with measurements in the SOL of tokamak plasmas [12, 16, 18].

The interpretation of these results is evident. For short waiting times and long burst duration, the signal Φ will at any time be influenced by many individual bursts, resulting in a large mean value and small relative variation. In the opposite limit of long waiting times and short duration, the signal is dominated by isolated burst events, resulting in a smaller mean value and large relative fluctuations, skewness and flatness. This is clearly illustrated in Fig. 1, which shows two numerical examples of the shot noise process given by Eq. (1) for exponentially distributed burst waiting times and amplitudes.

The results presented above show that the skewness and flatness vanish in the limit of large γ . It can be demonstrated that the probability density function for Φ then approaches a normal distribution [23, 24]. The distribution P_Φ can be written in terms of the characteristic function given in Eq. (7),

$$P_\Phi(\Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp\left[-i\Phi u + \sum_{n=1}^{\infty} \frac{(iu)^n \kappa_n}{n!}\right]. \quad (17)$$

In the limit of small τ_w/τ_d the exponential function can be expanded as a power series in u . Integrating term by term then gives

$$\Phi_{\text{rms}} P_\Phi(\Phi) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{\hat{\Phi}^2}{2}\right) \left[1 + \frac{\mu_3}{3!\Phi_{\text{rms}}^3(2\pi)^{1/2}}(\hat{\Phi}^3 - 3\hat{\Phi}) + R(\hat{\Phi})\right], \quad (18)$$

where R is the sum of the remaining terms in the expansion and the centred and rescaled amplitude is defined by $\hat{\Phi} = (\Phi - \langle \Phi \rangle) / \Phi_{\text{rms}}$. The terms inside the square bracket in Eq. (18) are of order 1, $(\tau_w / \tau_d)^{1/2}$ and τ_w / τ_d , respectively. This result shows how the probability density function for Φ approaches the normal distribution in the limit of large γ . This transition to normal distributed fluctuations is expected from the central limit theorem, since in this case a large number of burst events contributes to Φ at any given time.

The asymptotic probability density function in the strong intermittency regime for small γ can be obtained by neglecting overlap of individual burst events. Considering first a single burst event $\phi(t) = A \exp(-t / \tau_d)$, the time dt spent between ϕ and $\phi + d\phi$ is given by $dt / d\phi = \tau_d / \phi$. Note that due to the assumed exponential wave form, the burst amplitude A does not enter this expression. The number of bursts with amplitude above Φ is given by the complimentary cumulative amplitude distribution function, which for an exponential distribution is $\exp(-\Phi / \langle A \rangle)$. The probability density function P_Φ is given by the proportion of time which $\Phi(t)$ spends in the range from Φ to $\Phi + d\Phi$. With the appropriate normalization, the asymptotic probability density function in the strong intermittency regime is thus given by

$$\lim_{\gamma \rightarrow 0} P_\Phi(\Phi) = \lim_{\gamma \rightarrow 0} \frac{1}{\Gamma(\gamma)} \frac{1}{\Phi} \exp\left(-\frac{\Phi}{\langle A \rangle}\right), \quad (19)$$

where we have defined the Gamma-function

$$\Gamma(\gamma) = \int_0^\infty d\phi \phi^{\gamma-1} \exp(-\phi), \quad (20)$$

which to lowest order is given by $1/\gamma$ in the limit of small γ . This probability density function has an exponential tail for large amplitudes but is inversely proportional to Φ for small amplitudes due to the long quite period between burst events in this strong intermittency regime.

The characteristic function for a sum of independent random variables is the product of their individual characteristic functions. Thus, the probability that a sum of K burst events ϕ_k lies in the range between Φ and $\Phi + d\Phi$ is given by

$$\frac{d\Phi}{2\pi} \int_{-\infty}^{\infty} du \exp(-i\Phi u) \prod_{k=1}^K \langle \exp(i\phi_k u) \rangle, \quad (21)$$

where the characteristic functions are averaged over the values of ϕ_k . For general amplitude distribution and burst wave forms,

$$\langle \exp(i\phi_k u) \rangle = \frac{1}{T} \int_0^T dt_k \int_{-\infty}^{\infty} dA P_A(A) \exp[iAu\psi(t - t_k)],$$

where T is the duration of the time interval under consideration. Since all the K characteristic functions in Eq. (21) are the same, the conditional probability P_K is given by

$$P_K(\Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp(-i\Phi u) \langle \exp(i\phi_k u) \rangle^K,$$

assuming the number of events K in a time interval T to be given. The probability density function for the amplitude Φ is given by summing over all K ,

$$P_\Phi(\Phi) = \sum_{K=0}^{\infty} P(K) P_K(\Phi), \quad (22)$$

where $P(K)$ is given by Eq. (3). The stationary probability density function for Φ is obtained by letting $T \rightarrow \infty$. Some elementary manipulations lead to the desired result,

$$P_\Phi(\Phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ -i\Phi u + \frac{1}{\tau_w} \int_{-\infty}^{\infty} dA P_A(A) \int_{-\infty}^{\infty} dt [\exp(iAu\psi(t)) - 1] \right\}. \quad (23)$$

The logarithm of the characteristic function of P_Φ is thus

$$\frac{1}{\tau_w} \int_{-\infty}^{\infty} dA P_A(A) \int_{-\infty}^{\infty} dt [\exp(iAu\psi(t)) - 1] = \sum_{n=1}^{\infty} \frac{1}{\tau_w} \frac{(iu)^n}{n!} \int_{-\infty}^{\infty} dA A^n P_A(A) \int_{-\infty}^{\infty} dt [\psi(t)]^n. \quad (24)$$

where again the exponential function has been expanded. This establishes the general result stated by Eq. (5).

For the special case of exponentially distributed burst amplitudes, the amplitude integral in the above equation is given by $\langle A \rangle^n n!$, cancelling the factorial in Eq. (24). Further invoking the exponential wave form given in Eq. (11), it follows that the characteristic function for the stationary distribution can be written as

$$\exp \left[\gamma \sum_{n=1}^{\infty} \frac{(i\langle A \rangle u)^n}{n} \right] = (1 - i\langle A \rangle u)^{-\gamma}. \quad (25)$$

This is nothing but the characteristic function for a Gamma distribution with scale parameter $\langle A \rangle$ and shape parameter γ . Thus, the probability density function for Φ is given by

$$P_\Phi(\Phi) = \frac{1}{\langle A \rangle \Gamma(\gamma)} \left(\frac{\Phi}{\langle A \rangle} \right)^{\gamma-1} \exp \left(-\frac{\Phi}{\langle A \rangle} \right). \quad (26)$$

The lowest order moments and asymptotic limits of this distribution agree with the expressions discussed previously. In particular, the mean value is given by $\langle \Phi \rangle = \gamma \langle A \rangle$ and the variance by $\Phi_{\text{rms}}^2 = \gamma \langle A \rangle^2$.

For $\gamma > 1$, the most likely amplitude of the Gamma distribution is $(\gamma - 1)\langle A \rangle$ and the shape of the distribution function is unimodal and skewed. When $\gamma = 1$, P_Φ becomes an exponential distribution with the mean density given by the average burst amplitude,

$$P_\Phi(\Phi) = \frac{1}{\langle \Phi \rangle} \exp\left(-\frac{\Phi}{\langle \Phi \rangle}\right). \quad (27)$$

Note that by writing the average burst amplitude as $\langle A \rangle = \langle \Phi \rangle / \gamma$, the Gamma distribution given in Eq. (26) can be written in terms of the average plasma density as

$$\langle \Phi \rangle P_\Phi(\Phi) = \frac{\gamma}{\Gamma(\gamma)} \left(\frac{\gamma \Phi}{\langle \Phi \rangle}\right)^{\gamma-1} \exp\left(-\frac{\gamma \Phi}{\langle \Phi \rangle}\right), \quad (28)$$

where the scale parameter is given by $\langle \Phi \rangle / \gamma$ and the shape parameter is $\gamma = \langle \Phi \rangle^2 / \Phi_{\text{rms}}^2$. It should be noted that there are no fit parameter when comparing this prediction to experimental measurements. The above equation is exactly the form of the Gamma distribution found empirically to describe plasma fluctuations in the SOL of TCV across a broad range of plasma parameters [12]. Recently, it has also been shown to describe fluctuations recorded by gas puff imaging measurements in the SOL of the Alcator C-Mod tokamak [16].

In summary, a stochastic model for intermittent fluctuations in the boundary region of magnetized plasmas has been constructed as a random sequence of bursts which represent radial motion of blob-like structures. The mean plasma density is given by the average burst amplitude and the ratio of burst duration and waiting times. In the case of exponentially distributed burst amplitudes and waiting times, the amplitude probability density function is shown to by a Gamma distribution. This simple model thus explains the salient fluctuation statistics found in numerous experimental measurements and elucidates the role of burst statistics for large SOL plasma densities and fluctuation levels. The general parabolic relation between skewness and kurtosis moments predicted by this model likely explains the wide spread observation of this scaling relation in physical systems dominated by intermittent fluctuations.

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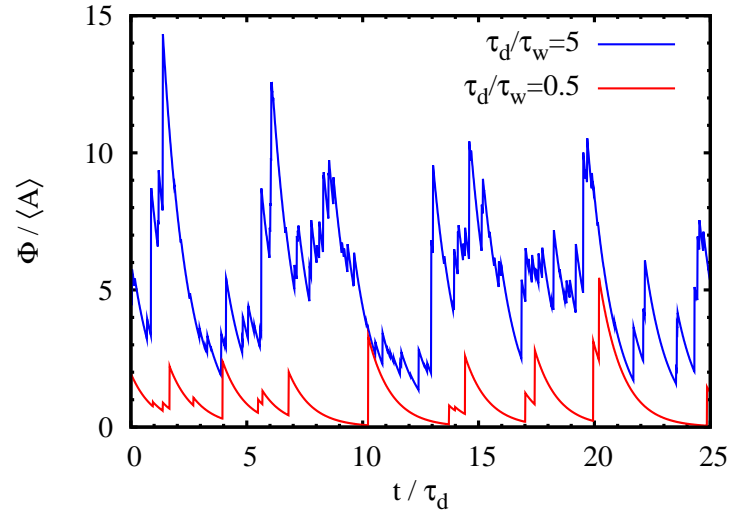


FIG. 1. Simulated shot noise time series for exponentially distributed burst amplitudes and waiting times.